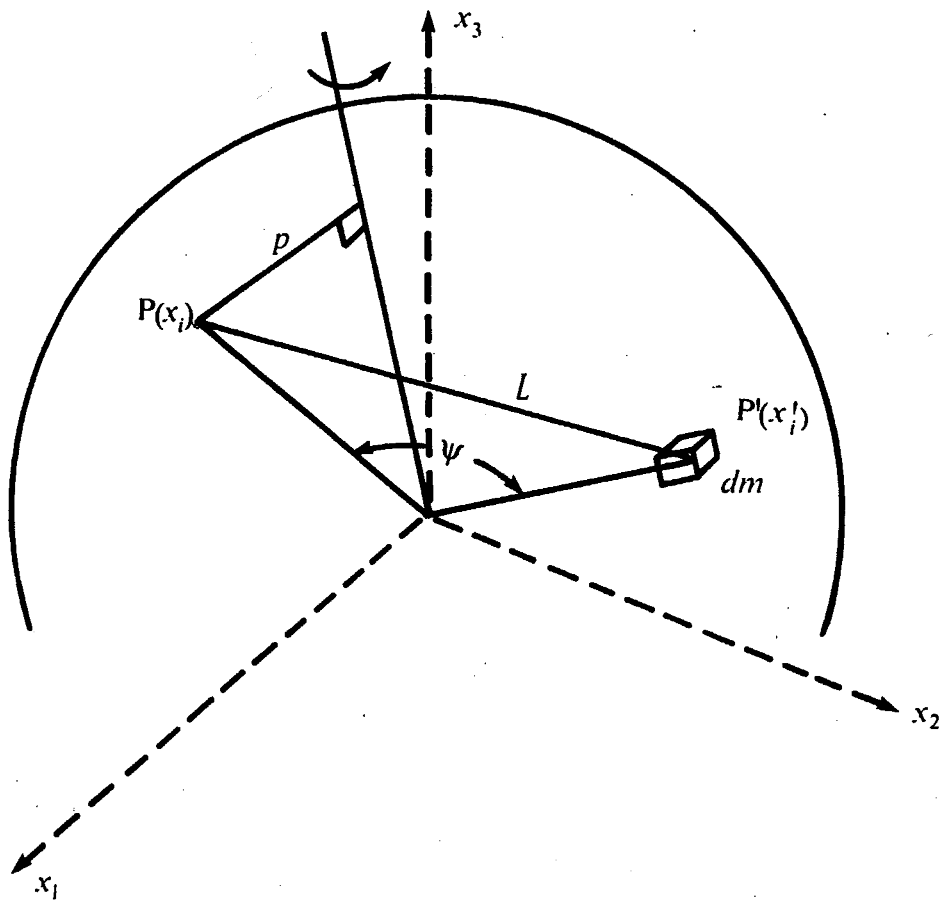

FIG. 2.1. Geometry of the relative positions of an element of mass dm, at P′(x_i'), of the Earth and the position P(x_i), at a distance L from P′, of a unit mass that is attracted by d$\mathcal{M}$. p is the distance of P from the instantaneous rotation axis $\boldsymbol{\omega}$.

is the corresponding distance, and ω is the rate of rotation about this axis (see Fig. 2.1). This axis is, for convenience, taken to be parallel to the x_3 axis, such that $p = (x_1^2 + x_2^2)^{1/2}$, and the components of this force parallel to the x_i axes are

$$(\omega^2 x_1, \omega^2 x_2, 0).$$

The position vector $\boldsymbol{L}$ in (2.1.6) is now defined relative to the rotating frame. The gravitational potential W of the combined attraction and centrifugal force is

$$\boldsymbol{g} = \nabla W = \nabla(V + \tfrac{1}{2}\omega^2 p^2). \qquad (2.1.7)$$

where $\boldsymbol{g}$ is the gravity vector whose magnitude g is called gravity.

2.2. Elements of potential theory

THE LAPLACE AND POISSON EQUATIONS

The gravitational potential V is defined by eqn (2.1.5) and the force components, the first spatial derivatives of V, are defined by eqns (2.1.3) and (2.1.4). Outside of the body, the second derivates of V satisfy the condition,

$$\sum_i \partial^2 V/\partial x_i^2 \equiv \nabla^2 V = 0, \tag{2.2.1}$$

and this is *Laplace's equation.* Its solutions are called *harmonic functions* and the gravitational potential is a harmonic function outside of the body. Within the body this same operation results in

$$\nabla^2 V = -4\pi G\rho \tag{2.2.2}$$

and this is known as *Poisson's equation.* The operator $\nabla^2 = \Sigma_i(\partial^2/\partial x_i^2)$ is called the Laplacian operator. While V and ∇V are continuous across the boundary containing the mass, $\nabla^2 V$ is not, and neither Laplace's or Poisson's equation is valid at the boundary itself. The potential W of the combined gravity and centrifugal forces does not satisfy these equations. From (2.1.7)

$$\nabla^2 W = \nabla^2 V + \nabla^2(\tfrac{1}{2}\omega^2 p^2) = 2\omega^2 \tag{2.2.3}$$

outside the body, and

$$\nabla^2 W = -4\pi G\rho + 2\omega^2 \tag{2.2.4}$$

within the body. In terms of spherical coordinates r, ϕ, λ (ϕ is latitude, λ is longitude, both defined relative to the Earth-fixed frame x_i)

$$x_i = r(\cos\phi\cos\lambda, \cos\phi\sin\lambda, \sin\phi)$$

and Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} - \frac{\tan\phi}{r^2}\frac{\partial V}{\partial \phi} + \frac{1}{r^2\cos^2\phi}\frac{\partial^2 V}{\partial \lambda^2} = 0. \tag{2.2.5}$$

LEGENDRE POLYNOMIALS

It is mathematically convenient to expand the external potential into harmonic functions because such functions are also solutions of Laplace's equation. Spherical harmonics are particularly convenient for representing observations made on the surface of a sphere, or at the Earth's surface, and they facilitate the geophysical inversions of global data sets. If the angle between the two radius vectors $\boldsymbol{r}$ of the unit mass at $P(x_i)$ and $\boldsymbol{r}'$ of the mass element at $P'(x_i')$ is denoted by ψ (Fig. 2.1), then the distance L between these two points is

$$L = (r^2 + r'^2 - 2rr'\cos\psi)^{1/2}. \tag{2.2.6}$$

For $r > r'$ the reciprocal distance is

$$L^{-1} = r^{-1}\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\psi\right]^{-1/2} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_{n0}(\cos\psi), \tag{2.2.7a}$$

and for $r < r'$

$$L^{-1} = \frac{1}{r'} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_{n0}(\cos\psi). \tag{2.2.7b}$$

The $P_{n0}(\cos\psi)$ are the conventional *Legendre polynomials* of degree n. They are defined by *Rodrigues'* formula, with $t = \cos\psi$, as (e.g. MacRobert 1967)

$$P_{n0}(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n. \tag{2.2.8}$$

The zero order polynomials P_{n0} are called *zonal harmonics*. They have n distinct zeros between $\phi = \pi/2$ and $-\pi/2$ arranged symmetrically about $\phi = 0$, and for odd n the circle $\phi = 0$ forms one of this set. If the positions of P and P′ are expressed in spherical coordinates r, ϕ, λ then the geocentric angle ψ is given by

$$\cos\psi = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda' - \lambda), \tag{2.2.9}$$

and substituting this into the above definition of the Legendre polynomial leads to the *addition theorem* (e.g. MacRobert 1967, p. 7),

$$P_{n0}(\cos\psi) = \sum_{m=0}^{n} (2 - \delta_{0m}) \frac{(n-m)!}{(n+m)!} P_{nm}(\sin\phi) \times P_{nm}(\sin\phi') \cos m(\lambda - \lambda') \tag{2.2.10}$$

where

$$\delta_{0m} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}. \tag{2.2.11}$$

The $P_{nm}(\sin\phi)$ are the *associated Legendre polynomials* of degree n and order m. They are defined by, now with $t = \sin\phi$,

$$P_{nm}(t) = \frac{1}{2^n n!} (1 - t^2)^{m/2} \frac{d^{n+m}}{dt^{n+m}} (t^2 - 1)^n, \tag{2.2.12a}$$

or, alternatively, by

$$P_{nm}(t) = \frac{(1 - t^2)^{m/2}}{2^n} \sum_{k=0}^{k^*} (-1)^k \frac{(2n - 2k)!}{k!\,(n-k)!\,(n-m-2k)!} t^{(n-m-2k)} \tag{2.2.12b}$$

where k^* is the greatest integer $\leqslant (n - m)/2$ (Heiskanen and Moritz 1967, p. 24). The polynomials $P_{nm}(\sin\phi)(\cos m\lambda$ or $\sin m\lambda)$ with $0 < m < n$ are caled *tesseral harmonics*. These functions have zeros along $n-m$ circles whose pole is $\phi = \pi/2$ and along m equally spaced great circles passing through $\phi = \pi/2$. For $m = n$ the polynomials are called *sectorial harmonics* but frequently the name tesseral harmonics is used to include all

Table 2.1
Unnormalized Legendre and associated Legendre functions $P_{nm}(\sin\phi)$ of degree n, order m.

	$m=0$	$m=1$	$m=2$	$m=3$
$n=0$	1			
$n=1$	$\sin\phi$	$\cos\phi$		
$n=2$	$\frac{3}{2}\sin^2\phi-\frac{1}{2}$	$3\sin\phi\cos\phi$	$3\cos^2\phi$	
$n=3$	$\frac{5}{2}\sin^3\phi-\frac{3}{2}\sin\phi$	$\cos\phi(\frac{15}{2}\sin^2\phi-\frac{3}{2})$	$15\cos^2\phi\sin\phi$	$15\cos^3\phi$

$m\neq 0$ harmonics, irrespective of their order. Table 2.1 summarizes some of the low degree and order functions.

With (2.2.7a) and (2.2.10) the potential V of (2.1.5) at $r>R$ can be expanded into spherical harmonics as

$$V=\frac{G\mathcal{M}}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n(C_{nm}\cos m\lambda+S_{nm}\sin m\lambda)P_{nm}(\sin\phi)\quad(2.2.13)$$

where

$$\left.\begin{matrix}C_{nm}\\ S_{nm}\end{matrix}\right\}=\frac{1}{\mathcal{M}R_e^n}(2-\delta_{0m})\frac{(n-m)!}{(n+m)!}\int_{\mathcal{M}}(r')^nP_{nm}(\sin\phi')\left\{\begin{matrix}\cos m\lambda'\\ \sin m\lambda'\end{matrix}\right\}\mathrm{d}\mathcal{M}.\quad(2.2.14a)$$

The C_{nm}, S_{nm} are the *Stokes coefficients*. R_e refers here to the equatorial radius of the planet. Sometimes the mean radius R is used in eqn (2.2.13) instead of R_e and in this case the definition (2.2.14a) of the coefficients must be correspondingly modified. That is,

$$\left.\begin{matrix}C_{nm}\\ S_{nm}\end{matrix}\right\}=\frac{1}{\mathcal{M}R^n}(2-\delta_{0m})\frac{(n-m)!}{(n+m)!}\int_{\mathcal{M}}(r')^nP_{nm}(\sin\phi')\left\{\begin{matrix}\cos m\lambda'\\ \sin m\lambda'\end{matrix}\right\}\mathrm{d}\mathcal{M}.\quad(2.2.14b)$$

With the abbreviations

$$Y_{inm}=P_{nm}(\sin\phi)\begin{cases}\cos m\lambda & \text{if } i=1\\ \sin m\lambda & \text{if } i=2\end{cases}\quad(2.2.15a)$$

$$C_{inm}=\begin{cases}C_{nm} & \text{if } i=1\\ S_{nm} & \text{if } i=2\end{cases},\quad(2.2.15b)$$

the potential is written in the abbreviated form

$$V(r,\phi,\lambda)=\frac{G\mathcal{M}}{r}\sum_{i=1}^{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^nC_{inm}Y_{inm}.\quad(2.2.15c)$$

SOME PROPERTIES OF LEGENDRE POLYNOMIALS

The Legendre polynomials have several important properties, foremost of which are the *orthogonality relations*,

$$\int_S Y_{inm} Y_{jpq} \, dS = 0 \tag{2.2.16a}$$

when $i \neq j$, $n \neq p$ or $m \neq q$, and

$$\int_S [Y_{inm}]^2 \, dS = 4\pi / \Pi_{nm}^2 \tag{2.2.16b}$$

where the integrals are over the surface S of a sphere of unit radius (e.g. MacRobert 1967). The normalizing factor Π_{nm} is defined by

$$\Pi_{nm}^2 = (2 - \delta_{0m})(2n+1)(n-m)!/(n+m)!. \tag{2.2.16c}$$

The definition (2.2.12) of the Legendre polynomials corresponds to the *unnormalized* functions frequently used in theoretical expansions of the potential. This usage does have the numerical disadvantage that as the degree and order increases, the term $(n+m)!$ in the denominators of eqn (2.2.14) becomes increasingly larger and to avoid this, *normalized* polynomials $\bar{Y}_{inm}$ are sometimes introduced. These are defined such that

$$\int_S [\bar{Y}_{inm}]^2 \, dS = 4\pi, \tag{2.2.16d}$$

or

and

$$\begin{aligned} \bar{P}_{inm} &= \Pi_{nm} P_{inm} \\ \bar{Y}_{inm} &= \Pi_{nm} Y_{inm}. \end{aligned} \tag{2.2.17a}$$

The corresponding *normalized Stokes coefficients* (2.2.14) are

$$\bar{C}_{inm} = \frac{1}{\Pi_{nm}} C_{inm}. \tag{2.2.17b}$$

The Y_{inm} defined by (2.2.15a) are *surface harmonics* and the products (r^n or $r^{-(n+1)}$)Y_{inm} are referred to as *solid spherical harmonics*. The latter are solutions of Laplace's equation, as is readily verified by substituting them into eqn (2.2.5). It also follows from (2.2.5) that

$$\left(\frac{\partial^2}{\partial \phi^2} - \tan \phi \frac{\partial}{\partial \phi} + \frac{1}{\cos^2 \phi} \frac{\partial^2}{\partial \lambda^2}\right) Y_{inm} = -n(n+1) Y_{inm}. \tag{2.2.18}$$

In some problems certain infinite sums of Legendre polynomials occur which can be expressed by analytical functions. These include

(Hobson 1932; Farrell 1972);

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty} P_n(\cos\psi) = \frac{1}{2\sin(\psi/2)}\\
&\sum_{n=0}^{\infty} nP_n(\cos\psi) = \frac{-1}{4\sin(\psi/2)}\\
&\sum_{n=1}^{\infty} \frac{\partial P_n(\cos\psi)}{\partial\psi} = \frac{-\cos(\psi/2)}{4\sin^2(\psi/2)}\\
&\sum_{n=1}^{\infty} \frac{1}{n}\frac{\partial P_n(\cos\psi)}{\partial\psi} = -\frac{\cos(\psi/2)[1+2\sin(\psi/2)]}{2\sin(\psi/2)[1+\sin(\psi/2)]}.
\end{aligned}\right\} \tag{2.2.19}$$

STOKES COEFFICIENTS

The Stokes coefficients (2.2.14a) represent integrals of functions of the mass distribution within the planet. For zero order, the S_{n0} vanish and the remaining coefficients C_{n0} are referred to as zonal coefficients. For degree 0

$$C_{00} = \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\,\mathrm{d}\mathcal{M} = 1$$

and the first term in the potential (2.2.13) is simply $G\mathcal{M}/r$, the potential at r caused by a radially symmetric sphere of mass $\mathcal{M}$. For degree 1,

$$\begin{aligned}
C_{10} &= \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\sin\phi'\,\mathrm{d}\mathcal{M} \equiv \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} x_3'\,\mathrm{d}\mathcal{M}\\
C_{11} &= \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\cos\phi'\cos\lambda'\,\mathrm{d}\mathcal{M} \equiv \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} x_1'\,\mathrm{d}\mathcal{M}\\
S_{11} &= \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\cos\phi'\sin\lambda'\,\mathrm{d}\mathcal{M} \equiv \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} x_2'\,\mathrm{d}\mathcal{M}
\end{aligned} \tag{2.2.20}$$

and these three coefficients represent the coordinates of the centre of mass of the body (normalized by R_e). They vanish if the origin of the coordinate system x_i is located at the centre of mass. The potential (2.2.13) can therefore be written as $V = V_0 + \Delta V$ where $V_0 = G\mathcal{M}/r$ and

$$\Delta V(r,\phi,\lambda) = \frac{G\mathcal{M}}{r}\sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n C_{inm}Y_{inm}. \tag{2.2.21a}$$

The second degree zonal Stokes coefficients follow from (2.2.14) and Table 2.1 as

$$C_{20} = \frac{-1}{\mathcal{M}R_e^2}[I_{33} - \tfrac{1}{2}(I_{11}+I_{22})], \tag{2.2.22a}$$

where

$$I_{ii} = \int_{\mathcal{M}} (x_{i+1}^2 + x_{i+2}^2)\, d\mathcal{M} \quad (i = 1, 2, 3) \qquad (2.2.23a)$$

represent the *moments of inertia* with respect to the x_i axes. Likewise

$$C_{21} = \frac{I_{13}}{\mathcal{M}R_e^2}, \qquad S_{21} = \frac{I_{23}}{\mathcal{M}R_e^2},$$
$$C_{22} = \frac{1}{4\mathcal{M}R_e^2}(I_{22} - I_{11}), \qquad S_{22} = \frac{I_{12}}{2\mathcal{M}R_e^2}, \qquad (2.2.22b)$$

with

$$I_{ij} = \int_{\mathcal{M}} x_i x_j\, d\mathcal{M}. \qquad (2.2.23b)$$

If R is used in (2.2.13) instead of R_e then the above expressions (2.2.20) to (2.2.23) must be modified accordingly.

To a good approximation the mean position of the Earth's rotation axis lies close to the mean position of the axis of maximum inertia and x_3 lies close to a principal axis. The $I_{13}/\mathcal{M}R_e^2$ and $I_{23}/\mathcal{M}R_e^2$ will therefore be small quantities in most cases and the corresponding potential to degree 2 is

$$V_2 \simeq \frac{G\mathcal{M}}{r} + \frac{G}{2r^3}[I_{33} - \tfrac{1}{2}(I_{11} + I_{22})](1 - 3\sin^2\phi)$$
$$+ \frac{3}{4}\frac{G}{r^3}[(I_{22} - I_{11})\cos 2\lambda + I_{12}\sin 2\lambda]\cos\phi. \qquad (2.2.21b)$$

Theoretical considerations of a rotating, fluid-like body indicates that the density distribution of the body will be symmetrical about the rotation axis so that $I_{12}/\mathcal{M}R_e^2$ and $(I_{11} - I_{22})/\mathcal{M}R_e^2$ can also be expected to be small quantities. The dominant Stokes coefficient of degree 2 will then be C_{20}, a measure of the Earth's flattening and which, for a fluid with the same mass, density distribution, and angular velocity as the Earth, will be of the order 10^{-3} (see below). This is indeed observed for the Earth, with (Gaposchkin 1977; Lerch *et al.* 1979)

$$\left.\begin{aligned} C_{20} &= -1082.63 \times 10^{-6} \\ C_{21}, S_{21} &= \mathcal{O}(10^{-9}) \\ C_{22} &= 1.57 \times 10^{-6}, \qquad S_{22} = -0.90 \times 10^{-6}. \end{aligned}\right\} \qquad (2.2.24)$$

$\mathcal{O}(x)$ refers to terms of quantities of the order of magnitude of x. These second degree Stokes coefficients are defined here with respect to the mean equatorial radius R_e (eqn 2.2.14a) rather than the mean radius R. All other coefficients are of the order $(C_{20})^2$ or smaller. For slowly rotating planets such as Venus, Mercury, or the Moon, C_{20} is considerably smaller and it need not necessarily be the dominant term in the corresponding gravitational potential expansion.

THE GEOID

Surfaces of constant gravitational potential, $W(x) = \text{constant} = W_0$, are called *equipotential surfaces* or *level surfaces*. The difference in potential $\mathrm{d}W$ between two nearby points separated by a distance $\mathrm{d}\boldsymbol{x}$ is

$$\mathrm{d}W = \sum_i \frac{\partial W}{\partial x_i}\,\mathrm{d}x_i = \nabla W \cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{x}$$

and if the vector $\mathrm{d}\boldsymbol{x}$ lies along W_0 then $\mathrm{d}W = \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{x} = 0$. The gravity vector $\boldsymbol{g}$ is therefore orthogonal to the equipotential surface passing through the same point and plumb lines or verticals are perpendicular to the level surfaces that they intersect.

In the absence of dynamical forces (winds, currents, for example), the ocean surface is a level surface of potential W_0. The ocean, therefore, provides a natural definition for the shape of the Earth, in particular as a number of geodetic measurements relate directly to this surface. Heights, measured by spirit levelling, are measured relative to the geoid (Chapter 5) and the radar altimeter measurements of the sea surface from satellites provides a nearly direct estimate of the shape of this surface (Chapter 6). This equipotential surface is called the *geoid*. It will lie partly within the Earth and the surface has to be extended mathematically to the continental areas. There the geoid does not have a unique definition and it is a function of the density distribution within the crust (Chapter 5).

Outside the Earth the equipotential surfaces are everywhere defined. Lines intersecting these surfaces perpendicularly specify the direction of the gravity vector, or the vertical or direction of the plumb line. Heights measured along these verticals with respect to the geoid are the *orthometric heights* discussed further in Chapter 5.

REFERENCE ELLIPSOID

The gravitational potential W at P follows from (2.1.7) and (2.2.13) as

$$W(r, \phi, \lambda) = \frac{G\mathcal{M}}{r}\left[1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\sin\phi)\right] + \tfrac{1}{3}\omega^2 r^2[1 - P_{20}(\sin\phi)] \qquad (2.2.25a)$$

because the distance of P from the rotation axis is

$$p^2 = r^2\cos^2\phi = \tfrac{2}{3}r^2[1 - P_{20}(\sin\phi)].$$

For the Earth C_{20} is the dominant term in the potential and a first approximation, U, of W is, at $r > R$,

$$U(r, \phi) = \frac{G\mathcal{M}}{r}\left\{1 + \frac{1}{3}\frac{\omega^2 r}{g_0(r)} + \left[C_{20}\left[\frac{R_e}{r}\right]^2 - \frac{1}{3}\frac{\omega^2 r}{g_0(r)}\right]P_{20}(\sin\phi)\right\} \quad (2.2.25b)$$

where $g_0(r) = G\mathcal{M}/r^2$. The shape of the equipotential surface corresponding to U is a function of latitude only and can be written in the form

$$r(\phi) = R[1 - \tfrac{2}{3}fP_{20}(\sin\phi)] + \mathcal{O}(f^2) \quad (2.2.26a)$$

with $f = (R_e - R_p)/R_e$ where R_e and R_p are the equatorial and polar radii and R is the mean radius. Equation (2.2.26a) represents the equation for an ellipsoid of revolution and, as a first approximation, the Earth's shape can be approximated by such a figure whose short axis coincides with the rotation axis. A relation between f and C_{20} follows by evaluating U at the equator $U(r = R_e, \phi = 0)$ with U at the pole $U(r = R_p, \phi = \pi/2)$, the two values being, by definition, on the same equipotential surface. The result is

$$-C_{20} = \tfrac{2}{3}f(1 - \tfrac{1}{2}f) - \tfrac{1}{3}m(1 - \tfrac{3}{2}m - \tfrac{2}{7}f) + \mathcal{O}(f^3) \quad (2.2.26b)$$

where

$$m = \frac{\text{centrifugal force at the equator}}{\text{gravity at the equator}} = \frac{\omega^2 R_e}{\gamma_e} \simeq 3 \times 10^{-3}.$$

The mean radius R in (2.2.26a) relates to the equatorial radius R_e by

$$R = R_e\left(1 - \frac{f}{3} - \frac{f^2}{5} + \ldots\right) \quad (2.2.27a)$$

and the theoretical gravity at the equator is

$$\gamma_e = \frac{G\mathcal{M}}{R_e^2}(1 - f + \tfrac{3}{2}m - \tfrac{15}{14}mf)^{-1}. \quad (2.2.27b)$$

The theoretical gravity at any latitude ϕ is given by

$$\gamma = \gamma_e(1 + f_2\sin^2\phi - \tfrac{1}{4}f_4\sin^2 2\phi + \ldots) \quad (2.2.27c)$$

with

$$f_2 = -f + \tfrac{5}{2}m - \tfrac{17}{14}fm + \tfrac{15}{4}m^2 + \ldots$$
$$f_4 = -\frac{f^2}{2} + \tfrac{5}{2}fm + \ldots \quad (2.2.27d)$$

The theoretical gravity at small heights above the ellipsoid, γ_h, can be expanded as

$$\gamma_h = \gamma + \frac{\partial\gamma}{\partial h}h + \frac{1}{2}\frac{\partial^2\gamma}{\partial h^2}h^2 + \ldots$$

and with (2.2.27c)

$$\gamma_h = \gamma - \frac{2\gamma_e}{R_e}[1 + f + m + (-3f + \tfrac{5}{2}m)\sin^2\phi]h + \frac{3\gamma_e}{R_e^2}h^2. \qquad (2.2.27e)$$

Together, eqns (2.2.25), (2.2.26a), and (2.2.27) define the potential, shape, and gravity of the first-order approximation of the Earth (see Heiskanen and Moritz 1967, for details). These definitions involve four parameters f, R_e, γ_e, and ω that are determined from the geodetic observations of the shape, gravity, and rotation of the planet. They provide a convenient reference surface with respect to which all departures in geometrical shape or in gravity can be treated as small quantities. Departures $\mathcal{N}$ of the observed equipotential from the theoretical equipotential surface may amount to 100 m or more and this quantity can be measured with a precision $\sigma_{\mathcal{N}}$ approaching 10 cm. Thus $\sigma_{\mathcal{N}}/R \simeq 10^{-8}$, less than quantities of the order f^2 or fm. The Stokes coefficient C_{20} is measured with a similar precision, as is gravity, and a more complete theory for the reference ellipsoid relations must contain higher order terms, of the order f^3 and f^4. Also, relations such as (2.2.25b) must contain terms in C_{40} and C_{60} (see Hirvonen 1960; Lambert 1961; Heiskanen and Moritz 1967).

Geodetic practice is to adopt a set of parameters that gives the best ellipsoidal approximation to the geoid and whose potential at its surface equals that of the geoid. This choice minimizes the departures of the observed quantities such as g from the theoretical values and reduces the number of higher order terms in the theoretical expressions such as (2.2.27). Recent observations and analyses lead to the following values for the fundamental geodetic parameters (EOS 1983).

$$\begin{aligned}
G\mathcal{M} &= (39\,860\,044 \pm 1)10^7\ \mathrm{m^3\,s^{-2}} \\
C_{20} &= -(1\,082.629 \pm 1)10^{-9} \\
R_e &= (6\,378\,136 \pm 1)\ \mathrm{m} \\
f^{-1} &= 298.275 \pm 0.001 \\
\omega &= 7\,292\,115 \times 10^{-11}\ \mathrm{rad\,s^{-1}} \\
\gamma_e &= (978\,032 \pm 1)10^{-5}\ \mathrm{m\,s^{-2}} \\
m &= 0.00345.
\end{aligned} \qquad (2.2.28)$$

HYDROSTATIC EQUILIBRIUM

The best-fitting ellipsoidal approximation of the geoid has no physical meaning and a geophysically more useful reference figure is the hydrostatic equilibrium shape of a body whose mass, radial density distribution and rotation are the same as the observed values for the planet.